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CORRECTNESS OF RESEARSHING SIGNAL SPECTRUMS WITH NONMONOTONIC BASIS FUNCTION

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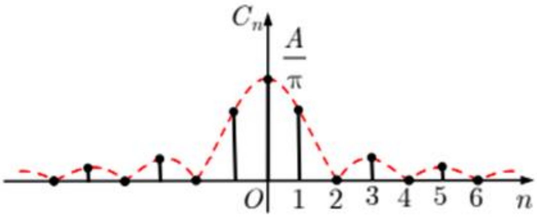
КОРЕКТНІСТЬ ДОСЛІДЖЕННЯ СПЕКТРІВ СИГНАЛУ З НЕМОНОТОННОЮ БАЗИСНОЮ ФУНКЦІЄЮ

Дана стаття має на меті наочно показати, як впливає дотримання необхідних умов, викладених в математичних теоремах на правильність отриманих результатів. Конкретно розглядатиметься вплив некоректного переносу інтегральної ознаки Коші на ряди спектрів сигналу на достовірність результатів.

Calculus has its share in various applied tasks. In particular, researching of signal spectrum is essential constituent of studying telecommunications theory: examining series made of spectrums is applied in modern digital signal processing. The most popular are spectrums with basis functions represented as a sine-wave.



For example, let's observe an amplitude spectrum of a signal represented by a periodic sequence of square-wave pulse:

$$C_n = \left|\frac{A}{\pi n}\sin\left(\frac{\pi n}{2}\right)\right|,$$

where A is amplitude.

This article is aimed to illustrate how adhering necessary conditions, which are posed in theorems, have its influence on correctness of obtained results. Therefore, perform examining a series with a summand equal to our original spectrum for convergence applying the integral test.

Integral Test. *Suppose that $f(x)$ is a continuous, positive, and decreasing function on the interval $[k, \infty)$ with $a_n = f(n)$. Then the following hold.*

1. *If* $\int_k^{\infty} f(x)\,dx$ *is convergent, then so is* $\sum_{n=k}^{\infty} a_n$.
2. *If* $\int_k^{\infty} f(x)\,dx$ *is divergent, then so is* $\sum_{n=k}^{\infty} a_n$.

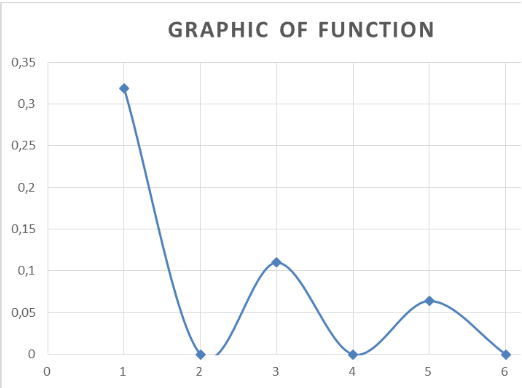
I write series of signal spectrums and examine for convergence:

$\sum_{n=1}^{\infty}\left|\frac{A}{\pi n}sin\frac{\pi n}{2}\right| = |A|\sum_{n=1}^{\infty}\left|\frac{1}{\pi n}sin\frac{\pi n}{2}\right| = |A|\left(\frac{1}{\pi}+0+\frac{1}{3\pi}+0+\frac{1}{5\pi}+\cdots\right) = |A|\sum_{n=0}^{\infty}\frac{1}{(2n+1)\pi}$,

$a_n = \frac{1}{2n+1} \sim \frac{1}{n} = b_n$

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n}$ – well-known harmonic series, which diverges, and so do $\sum_{n=0}^{\infty}\frac{1}{(2n+1)\pi}$.

Now, let's build function corresponding our original spectrum $f(x) = \left|\frac{A}{\pi}\right|\frac{\left|\sin\frac{\pi x}{2}\right|}{x}$. Obviously, our function is continuous, positive and decreasing (but not monotonically).



From other side I examine for convergence the integral $\left|\frac{A}{\pi}\right|\int_1^{\infty}\frac{\left|\sin\frac{\pi x}{2}\right|}{x}dx$. In general case integral like $\int_1^{\infty}\left|\frac{sin\,t}{t}\right|dt$ diverges, so we can affirm that so do $\int_1^{\infty}\frac{\left|\sin\frac{\pi x}{2}\right|}{x}dx$.

Conclusion 1: In this example relaxing the monotony condition of examining function in the integral test doesn't lead to incorrect result.

However, resemble relaxing frequently leads to mistakes during calculus. Now let's observe an illustrative case when incorrect transferring Cauchy integral test on random series leads to mistake. Accomplish this I make some alterations in examining spectrum:

$$C_n = \frac{A}{\pi n}(\sin^2(\frac{\pi n}{2}))^n$$

Series $\sum_{n=1}^{\infty}\frac{A}{\pi n}(\sin^2(\frac{\pi n}{2}))^n$ diverges analogically to the first example. Examine the improper integral $\left|\frac{A}{\pi}\right|\int_1^{\infty}\frac{(\sin\frac{\pi x}{2})^{2x}}{x}dx$ for convergence again.

$$\left|\frac{A}{\pi}\right|\int_1^{\infty}\frac{(\sin\frac{\pi x}{2})^{2x}}{x}dx = \left|\frac{A}{\pi}\right|\sum_{n=1}^{\infty}\int_n^{n+2}\frac{(\sin\frac{\pi x}{2})^{2x}}{x}dx \le \left|\frac{A}{\pi}\right|\sum_{n=1}^{\infty}\int_n^{n+2}\frac{(\sin\frac{\pi x}{2})^{2n}}{n}dx$$

Using changing of variables and property of the integral of periodic function and Wallis product get the next expression:

$$\int_{n}^{n+2} \frac{(\sin\frac{\pi x}{2})^{2n}}{n} dx = \frac{4}{\pi n}\int_{0}^{\frac{\pi}{2}}(\sin t)^{2n}\, dt = \frac{2(2n-1)!!}{n(2n)!!}$$

Now I have shown that original integral is smaller than original series. $\left|\frac{A}{\pi}\right|$ can be ignored because it is a numeric which I can shrink.

$$\int_{1}^{\infty} \frac{(\sin\frac{\pi x}{2})^{2x}}{x} dx \leq \sum_{n=1}^{\infty}\frac{2(2n-1)!!}{n(2n)!!}$$

Suppose $A_n = \frac{(2n-1)!!}{(2n)!!}$ and $B_n = \frac{(2n)!!}{(2n+1)!!}$. A step-by-step comparison shows that each factor in A_n is smaller than the corresponding factor in B_n,, hence $A_n < B_n$. So we have

$$A_n^2 \leq A_n B_n = \frac{1}{2n+1}$$

Returning to our infinite sum and employing the direct-comparison test,

$$\sum_{n=1}^{\infty}\frac{2(2n-1)!!}{n(2n)!!} = \sum_{n=1}^{\infty}\frac{2A_n}{n} \leq \sum_{n=1}^{\infty}\frac{2}{n\sqrt{2n+1}} \leq 2\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$$

Which is converges hence it is evident that in this case the researching result is incorrect.

Conclusion 2: In this example relaxing the monotony condition of examining function in the integral test leads to incorrect result.

Thus, having observed two illustrative examples of examining series for convergence applying Cauchy integral test it's easy to make sure that following posed in theorems conditions is exceedingly important for calculations, because relaxing even one cannot guarantee absolutely correct result.

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